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3 # TWO-DIMENSIONAL FLOW OF AN ELECTRICITY-CONDUCTING MEDIUM IN THE PRESENCE OF MASS FORCES AND OF A PLANE MAGNETIC FIELD 5

by

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[USSR]

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TWO-DIMENSIONAL FLOW OF AN ELECTRICITY-CONDUCTING MEDIUM IN THE PRESENCE OF MASS FORCES AND OF A PLANE MAGNETIC FIELD *

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SUMMARY

This paper considers all possible cases of flow of an ideal compressible conducting fluid under specific assumption. Some partial solutions of magnetohydrodynamics equations are obtained, which offer specific interest for the study of gradient winds and motions of cyclonic character in an electricity-conducting atmosphere.

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For large-scale motions in a conducting atmosphere (ionosphere, solar atmosphere) the effects of the force of gravity and the nonconservative deflection rotation force become determining. This is confirmed by the similitude principle, according to which the action of external forces is so much the greater as the scale of the event is greater. That is why the inclusion of these forces during theoretical investigations of large-scale events into the magnetohydrodynamics equations becomes indispensable.

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* DVUMERNOYE TECHENIYE ELEKTROPROVODNOY SREDY PRI NALICHII MASSOVYKH SIL I PLOSKOGO MAGNITNOGO POLYA

In the present work, we shall seek partial solutions of the magnetohydrodynamics equations for an ideal compressible conducting fluid in the presence of mass forces, when the velocity and the magnetic fields are given in the form :

$$u = u(y), \quad v = v(x), \quad w = 0; \tag{1}$$

$$H_x = H_x(y), \quad H_y = H_y(x), \ H_z = 0. \tag{2}$$

It is found that precisely such velocity and magnetic field forms provide the possibility of studying large-scale gradient winds and motions of a cyclonic character in a conducting atmosphere.

The system of magnetohydrodynamics equations has, at above-indicated assumptions, the form :

$$\operatorname{grad} P' = \rho(\mathrm{F} - (\mathrm{v}\nabla)\mathrm{v}) + (\mathrm{H}\nabla)\mathrm{H}/4\pi, \tag{3}$$

$$(\mathrm{v}\cdot\operatorname{grad}\rho) = 0, \tag{4}$$

$$(\mathrm{v}\nabla)\mathrm{H} - (\mathrm{H}\nabla)\mathrm{v} = 0, \tag{5}$$

where $P' = P + H^2/8\pi$ is the total pressure of the medium, ρ is the density, $\mathrm{F} = (2\lambda_3 v)\mathrm{i} + (-2\lambda_3 u)\mathrm{j} + (-g - 2\lambda_1 v)\mathrm{k}$ is the mass force, representing the sum of the force of gravity and of the deflecting rotation force; λ_3, λ_1 are the vector components of the angular velocity λ.

Below, we shall assume, everywhere, that the first derivatives from u, v, H_x and H_y always exist. The case, when one of these derivatives converts to zero, offers no significant interest, and that is why we shall not consider it. Thus, we postulate :

$$\frac{du}{dy} \neq 0, \quad \frac{dv}{dx} \neq 0, \quad \frac{dH_x}{dy} \neq 0, \quad \frac{dH_y}{dx} \neq 0. \tag{6}$$

In the coordinate form the equations (3), (4) and (5) take the form

$$\frac{\partial P'}{\partial x} = \rho\left(2\lambda_3 v - v\frac{\partial u}{\partial y}\right) + \frac{H_y}{4\pi}\frac{\partial H_x}{\partial y}, \tag{7}$$

$$\frac{\partial P'}{\partial y} = -\rho\left(2\lambda_3 u + u\frac{\partial v}{dx}\right) + \frac{H_x}{4\pi}\frac{\partial H_y}{\partial x}, \tag{8}$$

../..

$$\frac{\partial P'}{dz} = -\rho(g + 2\lambda_1 v), \tag{9}$$

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0, \tag{10}$$

$$v\frac{\partial H_x}{\partial y} - H_y\frac{\partial u}{\partial y} = 0, \tag{11}$$

$$u\frac{\partial H_y}{\partial x} - H_x\frac{\partial v}{\partial x} = 0. \tag{12}$$

Taking into account the condition (6), we may obtain from (11) and (12) the following equalities :

$$u = \alpha H_x, \quad v = \alpha H_y, \quad \vec{v} = \alpha\vec{H}, \tag{13}$$

where α is an arbitrary constant.

Eliminating the pressure P' from the equations (7)– (9), and taking into account the correlation (13), we shall have after simple transformations :

$$\left(2\lambda_3 + \frac{\partial v}{\partial x}\right)u\frac{\partial \rho}{\partial z} - (g + 2\lambda_1 v)\frac{\partial \rho}{\partial y} = 0, \tag{14}$$

$$2\lambda_1\rho\frac{\partial v}{\partial x} + (g + 2\lambda_1 v)\frac{\partial \rho}{\partial x} + \left(2\lambda_3 - \frac{\partial u}{\partial y}\right)v\frac{\partial \rho}{\partial z} = 0, \tag{15}$$

$$\rho\left(u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 v}{\partial y^2}\right) + \left(2\lambda_3 - \frac{\partial u}{\partial y}\right)v\frac{\partial \rho}{\partial y} + \left(2\lambda_3 + \frac{\partial v}{\partial x}\right)u\frac{\partial \rho}{\partial x} =$$

$$= \frac{1}{4\pi\alpha^2}\left(u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 u}{\partial y^2}\right). \tag{16}$$

Multiplying (14), (15) and (16 respectively by $\left(2\lambda_3 - \frac{\partial u}{\partial y}\right)v$, $-\left(2\lambda_3 + \frac{\partial v}{\partial x}\right)u$, $-g - 2\lambda_1 v$ and adding, we shall have as a result :

$$\rho\cdot\beta = \gamma, \tag{17}$$

where

$$\beta = 2\lambda_1\frac{\partial v}{\partial x}u\left(2\lambda_3 + \frac{\partial v}{\partial x}\right) - (g + 2\lambda_1 v)\left(u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 u}{\partial y^2}\right);$$

$$\gamma = \frac{1}{4\pi\alpha^2}(g + 2\lambda_1 v)\left(v\frac{\partial^2 u}{\partial y^2} - u\frac{\partial^2 v}{\partial x^2}\right);$$

The expression (17) may take place in one of the following cases:

1) either $\gamma = 0$, then β too must become zero, for in the opposite case the density ρ would also become zero;

2) or $\gamma \neq 0$, then $\beta \neq 0$.

1.- Let us investigate first of all the motion for the second case. Since $\gamma \neq 0$ and $\beta \neq 0$, the density ρ is determined directly from the equation (17):

$$\rho = \gamma / \beta = \rho(x, y). \tag{18}$$

Consequently, ρ does not depend on z. Then, from the equations (14), (6) and (10), we shall have: $\partial\rho / \partial y = 0$; $\partial\rho / \partial x = 0$. Therefore, in the case considered the medium must be viewed as incompressible. Taking into account the conditions (6), we shall find from the equation (15): $\lambda_1 = 0$, that is, the considered motion is possible only at the pole. Then the equations (16) and (18) become unconditional and we shall obtain for the density ρ:

$$\rho = \frac{1}{4\pi a^2}.$$

Substituting α from (19) into (13), we shall find:

$$v = H / \sqrt{4\pi\rho}. \tag{20}$$

In this case the equations (7) – (9) take the form:

$$\frac{1}{\rho}\frac{\partial P'}{\partial x} = 2\lambda_3 v, \quad \frac{1}{\rho}\frac{\partial P'}{\partial y} = -2\lambda_3 u, \quad \frac{1}{\rho}\frac{\partial P'}{\partial z} = -g. \tag{21}$$

Hence we determine the pressure:

$$P' = \frac{\lambda_3}{2\pi a^2}\int (v\,dx - u\,dy) - \frac{g}{4\pi a^2} z + P_0', \tag{22}$$

where P_0' is an arbitrary constant.

In the absence of mass forces ($\lambda_3 = 0$; $g = 0$), the obtained equation coincides with the well known stationary solution of magnetohydrodynamics [1, 2]. It may be shown that gradient wind must exist at the pole under the action of mass forces, that is, a wind blowing along the isobar. Indeed, multiplying the first of the equations (21) by $\partial P' / \partial y$, and the second by $-\partial P' / \partial x$ and adding, we obtain:

$$u\frac{\partial P'}{\partial x} + v\frac{\partial P'}{\partial y} = 0.$$

This equation expresses the condition of velocity vector perpendicularity to gradient of pressure. But, in the case of plane motion, the pressure gradient is the vector, directed along the normal to isobars.

Consequently, in the case under consideration the velocity vector is directed along the isobar.

Note that formula (21) provides the possibility of computing the gradient wind velocity by the gradient of pressure. For example, if we consider that the axis x is parallel to the wind direction, the second of the equations (21) may be rewritten in the form :

$$2V|\lambda_3| = \left|\frac{1}{\rho}\frac{\partial P'}{\partial n}\right|,$$

where V is the wind velocity, n is the direction of the normal to the isobar. Expressing, as an example for the ionosphere, λ_3 as the vertical component at the pole of the Earth's rotation vector through the value of this vector, and taking into account that at the pole the latitude $\varphi = 90^\circ$, we shall have :

$$|\lambda_3| = \lambda \sin\varphi = \lambda \quad \text{(for the Northern hemisphere)}$$

and consequently

$$V = G / 2\lambda, \qquad (23)$$

where G is the value of pressure gradient. Note that formula (23) can be generalized for an arbitrary latitude φ, provided we assume at the outset that the value of $-2\lambda_1\rho v$ in the initial equations (7) – (9) is small by comparison with the term $-\rho g$. In that case we shall obtain for the gradient wind :

$$V = G / 2\lambda \sin\varphi. \qquad (24)$$

The first two equations (21) in vectorial form may be written as follows :

$$\mathbf{F} + \mathbf{G} = 0,$$

where

$$\mathbf{F} = (2\lambda_3 v)\mathbf{i} + (-2\lambda_3 u)\mathbf{j}, \quad \mathbf{G} = \left(-\frac{1}{\rho}\frac{\partial P'}{\partial x}\right)\mathbf{i} + \left(-\frac{1}{\rho}\frac{\partial P'}{\partial y}\right)\mathbf{j},$$

whence it follows that the vectors **F** and **G** are equal in magnitude and have opposite directions. In the case of the ionosphere, taking into account that in the Northern hemisphere the deflecting force **F** is directed to the right , provided one looks in the direction **V** of the wind velocity, the vector **G** is directed to the left and since the pressure in the direction **G** varies from greater to lesser; thus, to the right of **V** we have a high pressure region and to the left — a low pressure region. The pattern is opposite in the Southern Hemisphere. This result coincides with the "Byuys - Ballo" law [3]*.

Therefore, the solution found for the case $\gamma \neq 0$; $\beta \neq 0$ points to the possibility of existence of gradient wind in a conducting atmosphere.

2. - Let us now pass to the investigation of motion in the case $\gamma = 0$, $\beta = 0$. The equating of γ and β to zero provides two conditions :

$$\lambda_1\left(2\lambda_3 + \frac{\partial v}{\partial x}\right) = 0, \quad \frac{1}{v}\frac{d^2 v}{dx^2} = \frac{1}{u}\frac{d^2 u}{dy^2}. \tag{25}$$

Two cases must be distinguished here 1) $\lambda_1 = 0$, 2) $2\lambda_3 + \partial v / \partial x = 0$.

Let us now examine at further length the first case : from the equations (14) — (16), taking into account the continuity equation (10), and $\lambda_1 = 0$ we may obtain the following equalities :

$$\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial \rho}{\partial x} = 0, \quad \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial \rho}{\partial y} = 0, \quad \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial \rho}{\partial z} = 0.$$

Hence: a) either $\partial \rho / \partial x = 0$, $\partial \rho / \partial y = 0$, or b) $\partial u / \partial y + \partial v / \partial x = 0$, $\partial \rho / \partial z = 0$.

Let us consider the case a). For u, v we shall have

$$u = -\Omega y + \xi, \quad v = \Omega x + \eta, \tag{26}$$

where Ω, ξ, η are arbitrary constants.

From the equation (13) we shall find for the magnetic field:

$$H_x = -ny + \xi_1, \quad H_y = nx + \eta_1, \tag{27}$$

where $n = \Omega / \alpha$, $\xi_1 = \xi / \alpha$, $\eta_1 = \eta / \alpha$. Assuming $\xi = \Omega q_2$, $\eta = -\Omega q_1$, where q_2, q_1 are other arbitrary constants, we shall obtain from (26) and (27)

$$\begin{gathered} u = -\Omega(y - q_2), \quad v = \Omega(x - q_1), \quad w = 0, \\ H_x = -n(y - q_2), \quad H_x = n(x - q_1), \quad H_z = 0. \end{gathered} \tag{28}$$

It follows from these formulas that the solution found represents the rotation of a conducting fluid, taking place at the pole with a constant angular velocity Ω about a center with coordinates $(x_0 = q_1;\ y_0 = q_2;\ z_0 = z)$.

* [in transliteration]

Substituting the values of u and v into the equations (14), (15), (16) and (10), we shall determine the density ρ:

$$\rho = \rho_0 e^{-\Phi(\sigma)}, \qquad (29)$$

where $\Phi(\sigma)$ is an arbitrary function of its argument;

$$\sigma = \frac{1}{2}\left\{(x-q_1)^2 + (y-q_2)^2 - 2\frac{g}{(\Omega+2\lambda_3)\Omega}z\right\}, \quad \rho_0 \cdot$$

Formula (29) shows that the density ρ varies with height; at the same time, by proper assortment of the arbitrary function $\Phi(\sigma)$ it is always possible to obtain that the density vary in a desirable fashion, for example, in correspondence with the true conditions in the conducting atmosphere.

Substituting ρ, u, v, H_x and H_y into the equations (7), (8) and (9), we shall find the pressure P' :

$$P' = \rho_0 \int \Psi(\sigma)\,d\sigma - \frac{\Omega^2}{8\pi\alpha^2}\{(x-q_1)^2 + (y-q_2)^2\} + P_0', \qquad (30)$$

where $\Psi(\sigma) = e^{-\Phi(\sigma)}, P_0'$ is a constant.

It follows from formulas (28) and (29) that the isobars constitute in every horizontal plane of current lines, just as do the magnetic lines of force, concentric circles with center at the point (x_0, y_0, z_0). Consequently, the solution found points to the possibility of motion of cyclonic character, taking place at the pole in a conducting atmosphere; at the same time in case of minimum, a stationary magnetohydrodynamic cyclone will be present at the center of isobars, and in case of maximum — an anticyclone (see [4]).

In the case b) the density is a constant quantity. For the determination of u and v, the single condition imposed to the velocity field, will be :

$$d^2v\,/\,dx^2 + \nu v = 0, \quad d^2u\,/\,dy^2 + \nu u = 0,$$

where ν is a constant.

At $\nu < 0$ we have the solution :

$$v = C_1 e^{\sqrt{-\nu}x} + C_2 e^{-\sqrt{-\nu}x}, \quad u = D_1 e^{\sqrt{-\nu}y} + D_2 e^{-\sqrt{-\nu}y}.$$

At $\nu = 0$

$$v = ax + b, \quad u = cy + d.$$

At $\nu > 0$

$$v = A_1 \cos\sqrt{\nu}x + A_2 \sin\sqrt{\nu}x, \quad u = B_1 \cos\sqrt{\nu}y + B_2 \sin\sqrt{\nu}y,$$

where $C_1, C_2, D_1, D_2, a, b, c, d, A_1, A_2, B_1, B_2$ are constants.

Depending upon the boundary conditions, only one solution for u and v may at all times be selected from the above solutions. Upon finding u and v, we shall determine H_x and H_y from the formulas (13), and the pressure P' from the equations (7), (8) and (9).

Let us now examine the second case when $\partial p / \partial x + 2\lambda_3 = 0$. Hence, $v = -2\lambda_3 x + \eta$, and then we shall find from (25) $u = ay + \xi$.

Assuming $a = -\Omega = +2\lambda_3$, $\xi = \Omega q_2$, $\eta = -\Omega q_1$, $n = \Omega / \alpha$, $\xi_1 = \xi / \alpha$, $\eta_1 = \eta / \alpha$, we shall obtain for the values of the velocity and of the magnetic field formulas analogous to (28)

$$u = -\Omega(y - q_2), \quad v = \Omega(x - q_1), \quad w = 0,$$
$$H_x = -n(y - q_2), \quad H_y = n(x - q_1), \quad H_z = 0,$$

but with the difference, however, that here Ω will no longer be an arbitrary constant, but a quantity linked with the latitude constant $\lambda_3 (-\Omega = 2\lambda_3)$, which, obviously, limits the motion quite considerably. In this case we find from (14) and (10), that $\partial \rho / \partial y = 0$, $\partial \rho / \partial x = 0$, and from the equation (15) we shall obtain $\lambda_1 = 0$. The equation (16) is fulfilled automatically. Consequently, ρ emerges as an arbitrary function of z:

$$\rho = \rho(z). \tag{31}$$

The pressure is found again by formulas (7), (8), (9):

$$P' = -\frac{\Omega^2}{8\pi\alpha^2}\{(x - q_1)^2 + (y - q_2)^2\} - \int_{z_0} \rho g \, dz + P_0'. \tag{32}$$

Hence it follows that in horizontal planes there are only "magnetic isobars", consisting of circles with center at the point (q_1, q_2, z_0).

This is the way we obtain a motion of cyclonic character, linked with the latitude constant φ. According to Fridman terminology [5], such motions are called "geoanticyclon" in standard meteorology.

Summing up, we conclude, that all possible cases of ideal compressible conducting fluid flow have been considered under the assumptions made at the outset.

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***** THE END *****

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